Resit Exam

12/04/2023, 8:30 am - 10:30 am

Instructions:

- Prepare your solutions in an **ordered, clear and clean way**. Avoid delivering solutions with scratches.
- Write your name and student number in **all** pages of your solutions.
- Clearly indicate each exercise and the corresponding answer. Provide your solutions with as much detail as possible.
- Use different pieces of paper for solutions of different exercises.
- Read first the whole exam, and make a strategy for which exercises you attempt first. Start with those you feel comfortable with!

Exercise 1: ($0.5 + 0.5 + 0.5$ points) Suppose that a surface $z = z(x, y)$ is implicitly defined by $F(x, y, z) = 0$. Assume that $(a, b, c) \in \mathbb{R}^3$ is a point in the surface, that is $F(a, b, c) = 0$.

a) Compute the direction along which the function $z(x, y)$ grows the fastest from the point $(x_0, y_0) = (a, b)$.
b) What is, geometrically, the set $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid z(x, y) = c\}$?
c) Find the equation of the tangent line to $\mathcal{C}$ at the point (a, b).

Solution:

a) According to Lecture 1, the direction of fastest change is given by the vector of directional derivatives, or the gradient. That is

$$\nabla F = \begin{bmatrix} \frac{\partial F}{\partial x}(a,b,c) \\ \frac{\partial F}{\partial y}(a,b,c) \\ \frac{\partial F}{\partial z}(a,b,c) \end{bmatrix}, \tag{1}$$

which is the gradient evaluated at the point of interest.

b) The set $\mathcal{C}$ is a one-dimensional (level) curve on the surface passing through (a, b, c).

c) The curve $\mathcal{C}$ is implicitly given by $F(x, y, c) = 0$. Therefore, we know from the lecture that the tangent line to $\mathcal{C}$ at (a, b, c) is given by $\ker \begin{bmatrix} \frac{\partial F}{\partial x}(a,b,c) \\ \frac{\partial F}{\partial y}(a,b,c) \end{bmatrix}$, which leads to a line given by

$$\frac{\partial F}{\partial x}(a,b,c)x + \frac{\partial F}{\partial y}(a,b,c)y = 0. \tag{2}$$

Exercise 2: (1 point) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \int_{h_1(x,y)}^{h_2(x,y)} g(t)\mathrm{d}t$, where $h_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, are differentiable functions and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Is f differentiable for all $(x, y) \in \mathbb{R}^2$? (Justify your answer)

Solution: By the fundamental theorem of calculus (keep in mind that the limits are functions) we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= g(h_2(x,y))\frac{\partial h_2}{\partial x} - g(h_1(x,y))\frac{\partial h_1}{\partial x} \\ \frac{\partial f}{\partial y} &= g(h_2(x,y))\frac{\partial h_2}{\partial y} - g(h_1(x,y))\frac{\partial h_1}{\partial y}. \end{aligned} \tag{3}$$

Since h_i is differentiable, their derivatives exist and are continuous. Since g is continuous and the product of continuous functions are continuous, we conclude that the partial derivatives of f are continuous, which implies differentiability of f in the whole plane.

Exercise 3: (1.5 points) Let $\mathcal{B} = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 \leq 1,\, x,y,z \geq 0\right\}$. Compute $\int_{\mathcal{B}} (x+y+z)|\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z|$.

Hint: you may want to use spherical coordinates.

Solution:

Spherical coordinates can be given by

$$S : (r,\theta,\phi) \mapsto (r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi) = (x,y,z) \tag{4}$$

and according to slide 15 of lecture 9 $\det|\mathrm{D}S| = r^2\cos\phi$. To parametrize $\mathcal{B}$ we let $r =\in [0,1]$ and $\theta \in [0,\pi/2]$, $\phi \in [0,\pi/2]$. In this way, and using slide 13 of lecture 9 and Fubini's theorem, we have:

$$\begin{aligned}
\int_{\mathcal{B}} (x+y+z)|\mathrm{d}x\mathrm{d}y\mathrm{d}z| &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (r\cos\theta\cos\phi + r\sin\theta\cos\phi + r\sin\phi)r^2\cos\phi\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^3(\cos\theta\cos^2\phi + \sin\theta\sin^2\phi + \sin\phi\cos\phi)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{1}{4}\int_0^{\pi/2}\int_0^{\pi/2} (\cos\theta\cos^2\phi + \sin\theta\sin^2\phi + \sin\phi\cos\phi)\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{1}{4}\int_0^{\pi/2} (\cos^2\phi + \sin^2\phi + \frac{\pi}{2}\sin\phi\cos\phi)\mathrm{d}\phi \\
&= \frac{1}{4}\left(\frac{\pi}{4} + \frac{\pi}{4} + \frac{\pi}{2}\cdot\frac{1}{2}\right) = \frac{3\pi}{16}.
\end{aligned} \tag{5}$$

Exercise 4: (1 + 0.25 + 0.25 points) Consider the ODE $x'' - 3x' + 2x = \sin(e^{-t})$.

a) Find the general solution of the given ODE.

b) Determine at least one initial condition for which $\lim_{t\to\infty} x(t) = +\infty$.

c) Determine at least one initial condition for which $\lim_{t\to\infty} x(t) = -\infty$.

Solution:

a) The characteristic polynomial for the homogeneous part is

$$s^2 - 3s + 2 = (s-1)(s-2), \tag{6}$$

meaning that the corresponding eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$. By making the substitution $y = (y_1, y_2) = (x, x')$ we obtain:

$$y' = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}}_{A} y + \underbrace{\begin{bmatrix} 0 \\ \sin(e^{-t}). \end{bmatrix}}_{b(t)} \tag{7}$$

Next we find the eigenvectors u, v of A associated to λ_1 and λ_2 respectively. We have for $\lambda_1 = 1$:

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\begin{bmatrix} u_2 \\ -2u_1 + 3u_2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{aligned} \tag{8}$$

leading to the eigenvector $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. On the other hand, for $\lambda_2 = 2$ we have:

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= 2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
\begin{bmatrix} v_2 \\ -2v_1 + 3v_2 \end{bmatrix} &= \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}
\end{aligned} \tag{9}$$

leading to the eigenvector $v = \begin{bmatrix}1\\2\end{bmatrix}$. So, the homogeneous solution reads as

$$y_h = c_1 e^t \begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\2\end{bmatrix} = \underbrace{\begin{bmatrix}e^t & e^{2t}\\ e^t & 2e^{2t}\end{bmatrix}}_{M(t)}\begin{bmatrix}c_1\\c_2\end{bmatrix}. \tag{10}$$

It follows that $M(t)^{-1} = \frac{1}{e^{3t}}\begin{bmatrix}2e^{2t} & -e^{2t}\\ -e^t & e^t\end{bmatrix} = \begin{bmatrix}2e^{-t} & -e^{-t}\\ -e^{-2t} & e^{-2t}\end{bmatrix}$. So $M(t)^{-1}b(t) = \begin{bmatrix}-e^{-t}\sin(e^{-t})\\ e^{-2t}\sin(e^{-t})\end{bmatrix}$. Let $u = e^{-t}$, then

$$\int_0^t -e^{-s}\sin(e^{-s})\mathrm{d}s = -\cos(e^{-s})|_0^t = -\cos(e^{-t}) + \cos(1) \tag{11}$$

and

$$\begin{aligned}\int_0^t e^{-2s}\sin(e^{-s})\mathrm{d}s &= -\int_1^{e^{-t}} u\sin(u)\mathrm{d}u = -(\sin(u) - u\cos(u))\Big|_1^{e^{-t}}\\ &= -(\sin(e^{-t}) - e^{-t}\cos(e^{-t})) + (\sin(1) - \cos(1))\end{aligned} \tag{12}$$

Since we only need a particular solution, we can disregard the constant terms. Hence:

$$y_p(t) = \begin{bmatrix}-\cos(e^{-t})\\ -(\sin(e^{-t}) - e^{-t}\cos(e^{-t}))\end{bmatrix} \tag{13}$$

and so the general solution reads as

$$y(t) = \begin{bmatrix}e^t & e^{2t}\\ e^t & 2e^{2t}\end{bmatrix}\left(\begin{bmatrix}c_1\\c_2\end{bmatrix} + \begin{bmatrix}-\cos(e^{-t})\\ -(\sin(e^{-t}) - e^{-t}\cos(e^{-t}))\end{bmatrix}\right) \tag{14}$$

From the last equation we can extract that

$$x(t) = c_1e^t + c_2e^{2t} - e^t\cos(e^{-t}) - e^{2t}\sin(e^{-t}) + e^t\cos(e^{-t}) = c_1e^t + c_2e^{2t} - e^{2t}\sin(e^{-t}). \tag{15}$$

It is convenient for the next two items to write:

$$x(t) = c_1e^t + e^{2t}(c_2 - \sin(e^{-t})). \tag{16}$$

b) If $c_1 > 0$ and $c_2 > 0$ then $\lim_{t\to\infty} x(t) = \infty$

c) If $c_1 < 0$ and $c_2 < 0$ then $\lim_{t\to\infty} x(t) = -\infty$

Exercise 5: (0.5 + 0.5 points) Consider the map $A : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $A(x, y) = (x^2\ln(xy), e^{x+y}, x^3y)$.

1. Compute the Jacobian matrix $\mathrm{D}A(1, 1)$.
2. Suppose that $B : \mathbb{R}^3 \to \mathbb{R}^3$ is a map whose Jacobian matrix at $(0, 1, 1)$ is $\mathrm{D}B(0, 1, 1) = \begin{bmatrix}-1 & 0 & 1\\ 0 & 2 & 2\end{bmatrix}$. Compute $\mathrm{D}(B \circ A)(1, 1)$.

Solution:

1.

$$\mathrm{D}A(x, y) = \begin{bmatrix}2x\ln(xy) + x & \frac{x^2}{y}\\ e^{x+y} & e^{x+y}\\ 3x^2y & x^3\end{bmatrix}, \tag{17}$$

therefore

$$\mathrm{D}A(1,1) = \begin{bmatrix} 1 & 1 \\ e^2 & e^2 \\ 3 & 1 \end{bmatrix}, \tag{18}$$

2. Correcting the map A to $A(x,y) = (x^2 \ln(xy), e^{x-y}, x^3 y)$ we have that

$$\mathrm{D}A(x,y) = \begin{bmatrix} 2x\ln(xy) + x & \frac{x^2}{y} \\ e^{x-y} & -e^{x-y} \\ 3x^2 y & x^3 \end{bmatrix}, \tag{19}$$

and so

$$\mathrm{D}A(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 1 \end{bmatrix}. \tag{20}$$

So, applying chain rule we have

$$\mathrm{D}(B \circ A)(1,1) = \mathrm{D}B(A(1,1))\mathrm{D}A(1,1) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 8 & 0 \end{bmatrix} \tag{21}$$

Exercise 6: (1.5 points) Let S be the part of the surface of equation $z = \sin xy + 2$ where

$$x^2 + y^2 \leq 1 \text{ and } x \geq 0,$$

oriented by the upward-pointing normal. Let $\vec{F} = \begin{bmatrix} 0 \\ 0 \\ x+y \end{bmatrix}$. What is the flux of $\vec{F}$ through S ?

Solution: Since the provided domain is a (part of a) disk, it is convenient to use polar coordinates to parameterise it. Let $U = (r\cos\theta, r\sin\theta)$ with $r \leq 1$ and $\theta \in [-\pi/2 \to \pi/2]$. Then $\gamma : (r,\theta) \mapsto (r\cos\theta, r\sin\theta, \sin(r^2\sin\theta\cos\theta) + 2)$ parameterizes S. To check if this is an orientation preserving, first notice that the normal to $F(x,y,z) = z - \sin xy - 2$ is given bu $n = \begin{bmatrix} -y\cos(xy) \\ -x\cos(xy) \\ 1 \end{bmatrix} = \begin{bmatrix} -r\sin\theta\cos(r^2\sin\theta\cos\theta) \\ -r\cos\theta\cos(r^2\sin\theta\cos\theta) \\ 1 \end{bmatrix}$. We also have

$$D_1\gamma = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 2r\sin\theta\cos\theta\sin(r^2\sin\theta\cos\theta) \end{bmatrix} \tag{22}$$

and

$$D_2\gamma = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ r^2\cos(r^2\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta). \end{bmatrix} \tag{23}$$

For simplicity let $u = \cos(r^2\sin\theta\cos\theta)$ and $v = \cos(r^2\sin\theta\cos\theta)$, so we now compute

$$\begin{aligned} \det \begin{bmatrix} -ur\sin\theta & \cos\theta & -r\sin\theta \\ -ur\cos\theta & \sin\theta & r\cos\theta \\ 1 & 2rv\sin\theta\cos\theta & ur^2(\cos^2\theta - \sin^2\theta) \end{bmatrix} &= 1(r\cos^2\theta + r\sin^2\theta) \\ &\quad - 2rv\sin\theta\cos\theta(-ur^2\sin\theta\cos\theta - ur^2\sin\theta\cos\theta) \\ &\quad + ur^2(\cos^2\theta - \sin^2\theta)(-ur\sin^2\theta + ur\cos^2\theta) \\ &= r + u^2r^3(\cos^2\theta - \sin^2\theta)^2 \geq 0. \end{aligned} \tag{24}$$

Therefore, the proposed parametrization is indeed orientation preserving. Finally, we can compute the flux as:

$$\begin{aligned}\int_S \Phi_{\vec{F}} &= \int_0^1 \int_{-\pi/2}^{\pi/2} \det \begin{bmatrix} 0 & \cos\theta & -r\sin\theta \\ 0 & \sin\theta & r\cos\theta \\ (r\cos\theta + r\sin\theta) & 2rv\sin\theta\cos\theta & ur^2(\cos^2\theta - \sin^2\theta) \end{bmatrix} \mathrm{d}\theta\mathrm{d}r \\ &= \int_0^1 \int_{-\pi/2}^{\pi/2} r^2(\cos\theta + \sin\theta)\mathrm{d}\theta\mathrm{d}r = 2\int_0^1 r^2\mathrm{d}r = \frac{2}{3}.\end{aligned} \tag{25}$$

Exercise 7: (1 point) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map given by $f(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$. Let $\phi = \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$. Show that

$$f^*\phi = \det(\mathrm{D}f)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Solution: We have:

$$f^*(\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n) = f^*\mathrm{d}y_1 \wedge \cdots f^*\mathrm{d}y_n = \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n, \tag{26}$$

where

$$\mathrm{d}f_i = \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k}\mathrm{d}x_k. \tag{27}$$

These two equations lead to the result using slide 13 of lecture 14. Arriving to the previous expression(s) is enough to get full points, and it is not necessary to show the previous sentence.

Exercise 8: (2 bonus points) Let S be a closed surface in $\mathbb{R}^3$ and V the solid that it encloses. Let S be oriented with the outward-pointing normal. Prove that $\mathrm{vol}_3\, V = \frac{1}{3}\int_S (x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y)$.